Energy decay in Burgers turbulence and interface growth: The problem of random initial conditions. II

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We present a study of the Burgers equation in one and two dimensions $d=1,2$ following the analytic approach indicated in a previous paper [S. E. Esipov and T. J. Newman, Phys. Rev. E **48**, 1046 (1993)]. For the problem of initial-condition decay we consider two classes of initial-condition distributions $Q_{1,2}\sim\exp[-(1/4D)\int(\nabla h)^2 d\mathbf{x}]$, where the h field is unbounded (Q_1) or bounded ($Q_2, |h|\leq H$). In one dimension these distributions give examples of nondegenerate and degenerate Burgers models of turbulence, respectively. Avoiding the replica trick and using an integral representation of the logarithm we study the exact analytically tractable field theory which has $d=2$ as a critical dimension. It is shown that the degenerate one-dimensional case has three stages of decay, when the kinetic-energy density diminishes with time as $t^{-2/3}$, t^{-2}, and $t^{-3/2}$, contrary to the predictions of the similarity hypothesis based on the second-order correlator of the distribution. In two dimensions we find the kinetic-energy decay which is proportional to $t^{-1}\ln^{-1/2}(t)$. It is shown that the pure diffusion equation with the Q_2-type initial condition has nontrivial energy decay exponents indicating connection with the O(2) nonlinear σ model.

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INTRODUCTION

The history of the Burgers equation [2] shows that the connection between Burgers turbulence and Navier-Stokes (NS) turbulence is a complicated issue. Citing the recent paper by Gotoh and Kraichnan [3], "by now it is clear that the differences between Burgers dynamics and NS dynamics are at least as significant as the similarities." This statement is made with respect to the one-dimensional version of the Burgers equation. It is also clear that the Burgers equation above one dimension has less relevance to real turbulence; for instance, the kinetic energy is not conserved in the inviscid limit. In its turn, the Burgers equation, being the simplest diffusive nonlinear equation, has a number of other physical applications to date. We may refer to the comparative usage in a renormalization-group (RG) study of NS equations [4], the field of interface growth in the framework of the Kardar-Parisi-Zhang (KPZ) equation [5], models of the large-scale structure of the Universe [6], solid-state-physics applications [7], etc. As in NS turbulence there are different means to excite the turbulent behavior. We think that a simpler problem to study is the relaxation of random initial conditions rather than the random-stirring (external noise) problem.

The present paper is a continuation of our recent work on Burgers turbulence and interface growth [1]. In Ref. [1] we have solved the problem for discrete initial conditions and indicated a field-theoretical approach suitable for continuous distributions. The field theory for the Burgers equation with continuous initial conditions resembles the so-called Liouville model [8] of strings and was applied to the nonbounded Gaussian distribution of initial conditions in one dimension. Studying the corresponding Schrödinger equation we essentially reproduced the Burgers result for the kinetic-energy decay [2] $E(t)\sim t^{-2/3}$.

The present work is partly devoted to the study of *degenerate* Burgers turbulence in one dimension and also contains an investigation of the two-dimensional case. We only study the decay of kinetic energy, the simplest local correlator. Our results in the degenerate $d=1$ case are in disagreement with the previously reported $E(t)\sim t^{-1}$ transients [6,9] whose derivation was based on the so-called similarity hypothesis involving the second-order correlator (Loitsyanskii correlator) of the distribution. The explanation is that the similarity hypothesis is based on dimensional arguments and in the degenerate case the Loitsyanskii correlator is zero by definition, whereas it is impossible to describe initial conditions without any parameter. With the distribution Q_2 we find that energy decays as $E(t)\sim t^{-2/3}$ as in the nondegenerate regime until the degeneracy becomes important at some crossover time. After that time the decay proceeds faster, $E(t)\sim t^{-2}$ for some time and (if viscosity is not equal to zero exactly) has a second crossover to the purely diffusive decay $E(t)\sim t^{-3/2}$.

Studying the Burgers equation it is natural to compare its behavior with and distinguish it from pure diffusion. This implies that for any of the cases considered it is the diffusion equation to be solved first. Consequently, as a by-product of the method of Green's functionals, we present the study of the diffusion equation in $d=2$ for the bounded distribution Q_2. The exponent for the energy-decay power law has a continuous dependence on the parameter D/H^2 (see below) up to a critical value $D/H^2=8/\pi$ when the exponent stops varying. This observation implies some connection with the O(2) nonlinear σ model [10,11].

The Burgers equation in two dimensions leads to functional equations for Green's functionals. The inviscid limit can be solved and the results show an explicit dependence upon short-distance cutoff. The kinetic energy decays as $E(t)\sim t^{-1}\ln^{-1/2}(t)$ for the Q_1 distribution. The onset of "boundness" is manifested by a sudden drop of the kinetic energy to zero (in the inviscid limit), thus showing some resemblance to the one-dimensional case. The reported results are beyond the reach of scaling arguments [12], which are helpful in the case of single power-law dependences.

I. THE BURGERS EQUATION AS A FIELD THEORY

We briefly rederive the field theory for the problem of the Burgers equation with random initial conditions in d dimensions for the sake of completeness. Consider the Burgers equation [2]

$$\partial_t \mathbf{v}=\nu\nabla^2\mathbf{v}-\tfrac{1}{2}\nabla v^2\ , \tag{1.1}$$

which with the help of the velocity potential $\nabla h=-\lambda^{-1}\mathbf{v}$ gives an equation

$$\partial_t h=\nu\nabla^2 h+\frac{\lambda}{2}(\nabla h)^2\ , \tag{1.2}$$

also known as the deterministic KPZ equation [5]. The parameter λ is introduced for convenience; it has dimensionality of length over time, and henceforth h has the dimensionality of length. The Hopf-Cole transformation to the new function $\exp(\lambda h/2\nu)$ results in a diffusion equation which is solved for a given (transformed) initial condition. The solution in terms of some initial h_0 reads

$$h(\mathbf{x},t)=\frac{2\nu}{\lambda}\ln\int d\mathbf{y}\,g(\mathbf{x}-\mathbf{y},t)\exp\frac{\lambda}{2\nu}h_0(\mathbf{y})\ , \tag{1.3}$$

where $g(\mathbf{x},t)=(4\pi\nu t)^{-d/2}\exp[-(x^2/4\nu t)]$ is the heat kernel (Green's function of the diffusion equation). The kinetic-energy density that we study here is defined as $E(t)=(\lambda^2/2)\langle(\nabla h)^2\rangle=\lambda\partial_t\langle h\rangle$, where we used translational invariance to drop the diffusion term. Using an integral representation of the logarithm in (1.3) we obtain

$$\langle h\rangle=\frac{2\nu}{\lambda}\int_0^\infty\frac{du}{u}[e^{-u}-\psi(u,t)] \tag{1.4}$$

and

$$E(t)=2\nu\partial_t\int_0^\infty\frac{du}{u}[e^{-u}-\psi(u,t)]\ , \tag{1.5}$$

where

$$\psi(u,t)=\int\mathcal{D}[h_0]\exp\{-S[h_0;u,t]\}\ . \tag{1.6}$$

This is written in the form of a field theory defined by the following action:

$$S[h_0;u,t]=\int d^dx\left\{\frac{1}{4D}(\nabla h_0)^2+ug(\mathbf{x},t)e^{\alpha h_0(\mathbf{x})}\right\}\ . \tag{1.7}$$

It resembles closely the Liouville model in string theory but differs by the heat kernel g, which makes the potential x dependent (i.e., "time" dependent). The one-dimensional case of this theory was considered in Ref. [1].

II. DEGENERACY OF BURGERS TURBULENCE

We are interested in different classes of distribution of random initial conditions which evolve unlike each other. We have had such an example in the case of discrete initial condition studied in Ref. [1]. In this section we present a comparison between bounded and unbounded Gaussian distributions of initial conditions and relate them to the degeneracy of the Burgers turbulence in one dimension.

The degeneracy of turbulence is related to the Loitsyanskii correlator D'. By analogy with the Navier-Stokes turbulence the Loitsyanskii correlator (in one dimension) is defined as [13,2,6,9]

$$D'=\frac{1}{2\lambda^2}\int_{-\infty}^{\infty}dx\,\langle v(x)v(0)\rangle\ . \tag{2.1}$$

It is known that the physics of (Burgers) turbulence is different depending on whether or not this correlator is zero. Usage of the velocity potential h makes it easier to understand the physical sense of D'. Indeed, take into account the adopted Gaussian initial conditions $Q_{1,2}$ given by

$$Q[h_0]\sim\exp\left[-\frac{1}{4D}\int(\nabla h_0)^2d^dx\right] \tag{2.2}$$

and consider the correlation of the velocity potential h following Burgers [2]. We have $h_0(x)-h_0(0)=-\lambda^{-1}\int_0^x v\,dx$. The two-point correlator of interest is given by

$$\begin{aligned}\langle[h_0(x)-h_0(0)]^2\rangle&=\lambda^{-2}\int_0^x\int_0^x v(x')v(x'')dx'dx''\\&=2\lambda^{-2}\int_0^x dz\,(x-z)\langle v(z)v(0)\rangle=2D'x+O(1)\ .\end{aligned} \tag{2.3}$$

Consequently all even correlators are $\langle[h_0(x)-h_0(0)]^{2n}\rangle=(2n-1)!!(2D'x)^n+O(x^{n-1})$ and Burgers concludes that the distribution of h at a given distance x is Gaussian, $P(h_2,h_1,x)=(4\pi D'|x|)^{-1/2}\exp[-(h_2-h_1)^2/4D'|x|]$. This speculation relies only on the fact that the problem is translationally invariant and the Loitsyanskii correlator D' is nonzero, while its integral converges quickly. We can see now that the nonbounded and bounded h distributions (2.2) belong to nonzero and zero D' distributions, respectively, and thus give representative examples of degenerate and nondegenerate Burgers turbulence. The reader may be interested in a direct calculation of D' using (2.2) which should give $D'=D$ for Q_1 and $D'=0$ for Q_2 distributions. As an example we present such a derivation for the Q_2 case in Sec. IV.

In two dimensions the distributions Q_1 and Q_2 are still different. The interface $h(\mathbf{x})$ is logarithmically rough (the distribution $P(h_2,h_1,x)$ is obtained in Sec. V]. We shall not make use of the Loitsyanskii correlator nor define the degeneracy above one dimension because of the lack of similarity with NS equations. Above two dimensions the h field is bounded by geometrical reasons and there is little difference between Q_1 and Q_2 distributions. The bound stems from the fact that the probability of finding the interface height h_2 at distance $\mathbf{x}$ from the given height h_1 is Gaussian

$$P(h_2,h_1,x)\sim \exp\left[-\frac{\pi^{d/2}a^{2-d}(h_2-h_1)^2}{D\Gamma(-1+d/2)(a^{4-2d}-x^{4-2d})}\right] . \tag{2.4}$$

a is the required lattice cutoff. The special case of a bound H which is *smaller* than the cutoff-related value $D^{1/2}a^{1-d/2}$ will not be considered in this paper.

III. THE DIFFUSION EQUATION: THE NONBOUNDED RANDOM INITIAL CONDITION

The solution of the diffusion equation corresponding to (1.2) for a given initial condition is

$$h(\mathbf{x},t)=\int d^dy\, g(\mathbf{x}-\mathbf{y},t)h_0(\mathbf{y}) . \tag{3.1}$$

For the nonbounded Gaussian distribution Q_1 of the type (2.2) one can calculate different correlators by using Fourier modes

$$h_0(\mathbf{x})=\frac{1}{(2\pi)^d}\int e^{i\mathbf{k}\cdot\mathbf{x}}h(\mathbf{k})d^dk ,$$

$$h(\mathbf{k})=\int e^{-i\mathbf{k}\cdot\mathbf{x}}h_0(\mathbf{x})d^dx .$$

The solution of the diffusion equation (3.1) is given by

$$h(\mathbf{k},t)=h_0(\mathbf{k})e^{-\nu k^2 t} . \tag{3.2}$$

In terms of Fourier modes the kinetic-energy density is

$$E(t)=\frac{\lambda^2}{2}\langle(\nabla h)^2\rangle=-\frac{\lambda^2}{2(2\pi)^{2d}}\int d^dk\, d^dk'kk'e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')}\langle h(\mathbf{k},t)h(\mathbf{k},t)\rangle$$

$$=-\frac{\lambda^2}{2(2\pi)^{2d}}\int d^dk\, d^dk'kk'e^{i\mathbf{x}\cdot(\mathbf{k}+\mathbf{k}')-\nu(k^2+k'^2)t}\langle h_0(\mathbf{k})h_0(\mathbf{k}')\rangle . \tag{3.3}$$

The functional integral

$$\langle h_0(\mathbf{k})h_0(\mathbf{k}')\rangle =\int \mathcal{D}[h_0]h_0(\mathbf{k})h_0(\mathbf{k}') \times\exp\left[-\frac{1}{(2\pi)^d 4D}\int k^2h_0(\mathbf{k})h_0(-\mathbf{k})d^dk\right] \tag{3.4}$$

equals

$$\langle h_0(\mathbf{k})h_0(\mathbf{k}')\rangle=2Dk^{-2}(2\pi)^d\delta(\mathbf{k}+\mathbf{k}') \tag{3.5}$$

and gives

$$E(t)=\frac{\lambda^2 D}{(8\pi\nu t)^{d/2}} . \tag{3.6}$$

IV. THE DIFFUSION EQUATION: THE BOUNDED RANDOM INITIAL CONDITION IN ONE DIMENSION

This is still a simple but algebraically consuming calculation which uses mirror imaging in treating bounded Gaussian initial conditions. The representation in terms of Fourier modes is less convenient and the calculation is performed in the original space. The connection between the kinetic-energy density and the correlator of the initial field in d dimensions is

$$E(t)=\frac{\lambda^2}{2}\langle(\nabla h)^2\rangle =\frac{\lambda^2}{2}\int\int d\mathbf{y}_1 d\mathbf{y}_2[\nabla g(y_1)\nabla g(y_2)] \times\langle h_0(\mathbf{y}_1)h_0(\mathbf{y}_2)\rangle . \tag{4.1}$$

Integrating over the half-sum $|\mathbf{y}_1+\mathbf{y}_2|$ and angles, we obtain the following expression:

$$E(t)=\frac{2d\lambda^2}{\Gamma(d/2)(8\nu t)^{1+d/2}} \times\int_0^\infty dr\, r^{d-1}e^{-r^2/8\nu t} \times\left[1-\frac{r^2}{4d\nu t}\right]\langle h_0(r)h_0(0)\rangle . \tag{4.2}$$

In one dimension the weight of a given "interface," or rather trajectory, which starts at some $h_0(0)=h_1$ and ends at $h_0(x)=h_2$,

$$\bar{G}(h_2,h_1;x_2,x_1) =\int_{h_0(x_1)=h_1}^{h_0(x_2)=h_2}\mathcal{D}[h_0]\exp\left[-\frac{1}{4D}\int_{x_1}^{x_2}\left[\frac{dh_0}{dy}\right]^2 dy\right] , \tag{4.3}$$

obeys the diffusion equation in (h,x) space [not to be confused with the original (x,t) space]

$$\partial_x G = \pm D \partial_{hh} G \ , \tag{4.4}$$

where $h = h_{1,2}$, $x = x_{1,2}$, and $\pm$ correspond to forward ($h = h_2$ and $x = x_2$) and backward ($h = h_1$ and $x = x_1$) equations, respectively. Equation (4.4) is useful for imposing the constraint $|h| \le H$ explicitly. Namely, we introduce boundary conditions $\partial_h G(\pm H, x) = 0$ of zero flux through the boundaries and consider the equation in the stripe $|h| \le H$. In order to solve the initial condition problem $G(h,0) = \delta(h - h_1)$, we regard the $\pm H$ boundaries as mirrors and sum the usual Green's functions of the unconstrained Eq. (4.4) over images

$$G_\Sigma(h_2,h_1,x) = \sum_{m=-\infty}^{\infty} [G(h_2 - h_1 + 4Hm, x) + G(h_2 + h_1 - 2H + 4Hm, x)] \ , \tag{4.5}$$

where

$$G(h,x) = (4\pi D x)^{-1/2} \exp(-h^2/4Dx) \ . \tag{4.6}$$

According to (4.2) one needs to evaluate the correlator $\langle h_0(x) h_0(0) \rangle$, which is given by

$$\langle h_0(x) h_0(0) \rangle = \frac{1}{2H} \int_{-H}^{H} \int_{-H}^{H} dh_1 dh_2 h_1 h_2 G_\Sigma(h_2,h_1,x) \ . \tag{4.7}$$

The factor $1/2H$ is due to averaging over possible values of h_1 which are clearly uniform in $[-H,H]$. Performing the elementary integration and making use of Appendix A we find

$$\langle h_0(x) h_0(0) \rangle = \frac{32H^2}{\pi^4} \sum_{n=0}^{\infty} \frac{\exp\left[-\frac{\pi^2 D |x| (2n+1)^2}{4H^2} \right]}{(2n+1)^4} \ . \tag{4.8}$$

In the integral (4.2), consider the late time asymptotic $t \gg H^4/D^2\nu$; then

$$E(t) = \frac{32\lambda^2 H^4}{\pi^5 D (2\pi \nu t)^{3/2}} \ . \tag{4.9}$$

The exact formula containing a sum over parabolic cylinder functions can be derived from (4.2) and (4.8) for arbitrary times. Summarizing, at short times $t \ll H^4/D^2\nu$, we have energy decay given by Eq. (3.6), $E(t) \propto t^{-1/2}$, and after the crossover the decay proceeds faster, $E(t) \propto t^{-3/2}$.

The correlator (4.8) allows the explicit calculation of the Loitsyanskii correlator (2.1). One finds

$$\langle v(x) v(0) \rangle = -\lambda^2 \partial_{xx} \langle h_0(x) h_0(0) \rangle = 2D\lambda^2 \delta(x) - \frac{2D^2\lambda^2}{H^2} \sum_{n=0}^{\infty} \exp\left[-\frac{\pi^2 D |x| (2n+1)^2}{4H^2} \right] . \tag{4.10}$$

The sum can be written by using the elliptic function θ_2. Performing the remaining integration over x in (2.1) we find that the Brownian δ correlator just cancels the nonlocal part generated by the $|h| \le H$ constraint, so that finally $D' = 0$. The geometrical interpretation of degeneracy leads naturally to the potential associated with fluctuations of h field. The Q_2 distribution corresponds to a potential well with infinite walls at $|h| = H$.

V. THE GREEN'S FUNCTIONAL IN TWO DIMENSIONS

We continue our preliminary study of diffusion in order to develop the method and have linear results for comparison with the Burgers equation. The derivation of the Green's functional is analogous to the oscillator problem in quantum mechanics [14,11]. We return to Eqs. (1.6) and (1.7) and consider a functional

$$\mathcal{F}[h_2(\phi), h_1(\phi); y_2, y_1] = \int_{h(\phi,y_1)=h_1(\phi)}^{h(\phi,y_2)=h_2(\phi)} \mathcal{D}[h] \exp\left[-\frac{1}{4D} \int_{y_1}^{y_2} y\, dy \int_0^{2\pi} d\phi (\nabla h)^2 \right] . \tag{5.1}$$

We prefer to use polar system of coordinates where "time" is the radius y. In what follows the adjective "functional" will be omitted in many cases.

The proper normalization of $\mathcal{F}$ will be specified later; we need to evaluate $\mathcal{F}$ first. Introducing Fourier modes for angular dependence

$$h(\phi) = \sum_{k=-\infty}^{\infty} h_k e^{ik\phi}, \quad h_k = \frac{1}{2\pi} \int_0^{2\pi} d\phi\, h(\phi) e^{-ik\phi} \tag{5.2}$$

and changing variables to real and imaginary parts $h_{\pm k} = \alpha_k \pm i\beta_k$, we rewrite (5.1) as

$$\mathcal{F}[h_2(\phi), h_1(\phi); y_2, y_1] = \prod_{k=0}^{\infty} I_k(\alpha_k) I_k(\beta_k) \ , \tag{5.3}$$

where massive one-dimensional theories are defined for each mode

$$I_k(\gamma) = \int_{\gamma_1}^{\gamma_2} \mathcal{D}[\gamma] \exp\left\{ -\frac{\pi}{(1+\delta_k) D} \times \int_{y_1}^{y_2} dy\, y \left[\left[\frac{\partial \gamma}{\partial y} \right]^2 + \frac{k^2 \gamma^2}{y^2} \right] \right\} , \tag{5.4}$$

where δ_k is the Kronecker symbol. To evaluate this Gaussian integral we first determine the classical trajectory defined by the Euler equation

$$y^2\gamma''+y\gamma'-k^2\gamma=0 . \tag{5.5}$$

This equation has the solution

$$\gamma=C_1y^k+C_2y^{-k} , \tag{5.6}$$

with two constants of integration to satisfy the boundary conditions $\gamma(y_1)=\gamma_1$ and $\gamma(y_2)=\gamma_2$. The solution is

$$\gamma(y)=\frac{\gamma_1y_1^k-\gamma_2y_2^k}{y_1^{2k}-y_2^{2k}}y^k+\frac{y_1^ky_2^k}{y^k}\frac{\gamma_2y_1^k-\gamma_1y_2^k}{y_1^{2k}-y_2^{2k}} . \tag{5.7}$$

The zeroth mode $k=0$ has a different trajectory,

$$\gamma(y)=\frac{\gamma_1\ln y_2-\gamma_2\ln y_1}{\ln\dfrac{y_2}{y_1}}+\frac{\gamma_2-\gamma_1}{\ln\dfrac{y_2}{y_1}}\ln y . \tag{5.8}$$

Calculating the classic action we get

$$-\frac{\pi k}{D}[(\gamma_1^2+\gamma_2^2)f_s-2\gamma_1\gamma_2f_c] \tag{5.9}$$

for nonzero k and

$$-\frac{\pi}{2D}\frac{(\gamma_2-\gamma_1)^2}{\ln\dfrac{y_2}{y_1}} \tag{5.10}$$

for the zeroth mode. Here the functions of stereographic projection are introduced

$$f_s=(r^2+1)/(r^2-1),\quad f_c=2r/(r^2-1) ,\quad r=(y_2/y_1)^k \tag{5.11}$$

and

$$f_s^2-f_c^2=1 . \tag{5.12}$$

Returning to the functional one gets

$$\mathcal{F}[h_2,h_1;y_2,y_1]=C\exp\left[-\frac{\pi(\alpha_{2,0}-\alpha_{1,0})^2}{2D\ln(y_2/y_1)}\right]\times\exp\left\{-\sum_{k=1}^{k=\infty}\frac{\pi k}{D}[(\alpha_{1,k}^2+\beta_{1,k}^2+\alpha_{2,k}^2+\beta_{2,k})f_s-2(\alpha_{1,k}\alpha_{2,k}+\beta_{1,k}\beta_{2,k})f_c]\right\} , \tag{5.13}$$

or, in terms of the original Fourier modes,

$$\mathcal{F}[h_2,h_1;y_2,y_1]=C\exp\left\{-\sum_{k=-\infty}^{\infty}\frac{\pi k}{D}[(h_{1,k}h_{1,-k}+h_{2,k}h_{2,-k})f_s-(h_{1,k}h_{2,-k}+h_{1,-k}h_{2,k})f_c]\right\} . \tag{5.14}$$

Note that the factors $kf_{s,c}$ are even in k and have proper limits, so that the summation in (5.14) is expanded to all integers, including $k=0$.

Integrating $\mathcal{F}$ over (say) the final field h_2, one finds that it is not normalized. The reason is that the representations (5.3) and (5.4) describe a massive theory and classic action becomes zero only if all $h_{1,k}=0$ for $k\neq0$. The zeroth mode can be independently normalized in Eq. (5.14). Other modes, if normalized forcibly, give rise to inconvenient nonlocal factors which do not ensure convolution properties. Thus one has to perform calculation with the functional $\mathcal{F}$ and normalize the answer for all nonzero modes with respect to the bare theory at the very end of the calculation. With this in mind $\mathcal{F}$ can be considered as the Green's *functional.*

We then derive the differential equation which the integrals (5.1) and (5.14) satisfy. In complete analogy with the one-dimensional case [14] we consider a small increment $y_2+\delta y$ and the corresponding variation of the field h_2:

$$\mathcal{F}[h_2+\eta,h_1;y_2+\delta y,y_1]=\int\mathcal{D}[\eta]\,\mathcal{F}[h_2+\eta,h_1;y_2,y_1]\exp\left\{-\frac{y_2}{4D\delta y}\int_0^{2\pi}d\phi\,\eta^2-\frac{\delta y}{4Dy_2^2}\int_0^{2\pi}d\phi\left[\frac{dh_2}{d\phi}\right]^2\right\} . \tag{5.15}$$

Expanding $\mathcal{F}[h_2+\eta,h_1;y_2,y_1]$ to second order in the variation η and integrating out η, one finds that the linear term in η vanishes and the second-order derivative is nonzero only if the arguments of the $h_2(\phi)$ functions are equal. Expanding also the δy dependence to the first order, we finally obtain

$$\partial_y\mathcal{F}=\frac{D}{y}\int_0^{2\pi}d\phi\frac{\delta^2\mathcal{F}}{\delta h^2}-\frac{\mathcal{F}}{4Dy}\int_0^{2\pi}d\phi\left[\frac{dh}{d\phi}\right]^2 , \tag{5.16}$$

where the subscript 2 can be omitted. We shall use the symbolic notation for the diffusion operator of closed strings and write Eq. (5.16) as

$$\partial_y\mathcal{F}=\hat{D}\mathcal{F} . \tag{5.17}$$

Clearly, the normalization condition is not satisfied: Eq. (5.16) contains a decay term. The solution of Eq. (5.16) is

$$\mathcal{F}[h_2,h_1;y_2,y_1]=C'\prod_k\left[\frac{kf_c}{2D}\right]^{1/2}\exp\left\{-\frac{\pi k}{2D}[(h_{1,k}h_{1,-k}+h_{2,k}h_{2,-k})f_s-(h_{1,k}h_{2,-k}+h_{1,-k}h_{2,k})f_c]\right\} , \tag{5.18}$$

where C' is now independent of h_1, h_2, y_1, y_2. This formula was found by T. J. Newman and it contains the explicit prefactor of the functional $\mathcal{F}$ (5.14). The prefactor can be alternatively found by computing Gaussian fluctuations around the classic trajectory. With the help of the prefactor the convolution property can be proved

$$\mathcal{F}[h_2,h_1;y_2,y_1]=\int \mathcal{D}[h_3]\mathcal{F}[h_2,h_3;y_2,y_3]\times\mathcal{F}[h_3,h_1;y_3,y_1] \; . \tag{5.19}$$

The following identities are useful:

$$\begin{aligned} f_c(y_3/y_1)f_c(y_2/y_3)[f_s(y_3/y_1)+f_s(y_2/y_3)] &= f_c(y_2/y_1) \; , \\ f_s^2(y_3/y_1)-f_s^2(y_2/y_3) &= f_c^2(y_2/y_1) \; . \end{aligned} \tag{5.20}$$

The Green's functional (5.18) is connecting angular $h(\phi)$ profiles between two nonzero radial times y_1 and y_2, $y_2>y_1$. In some cases we shall need to set $y_1=0$. The logarithmic dependence in (5.18) leads to a divergence here. Divergence indicates that the weight $(\nabla h)^2$ in the action is insufficient to ensure convergence of the functional. This problem is usually resolved by introducing the uv cutoff a. In a proper field theory it is anticipated that the cutoff present in bare values must be eliminated by renormalization when interaction is included. However, there is *no* reason to expect that the Burgers turbulence is a proper field theory.

In some cases we shall need the angular part of the Green's functional (5.18) to be integrated out. This happens, for example, when one calculates the two-point correlator $\langle h(x)h(0)\rangle$. Although this correlator does not formally exist in the unbounded case, it is useful to define the probability $P(h_2,h_1,x)$ of arriving at $h(x)=h_2$ provided that $h(0)=h_1$. It is given by the integral (to be normalized)

$$P(h_2,h_1,x)\sim\int \mathcal{D}[h_2(\phi)]\mathcal{F}[h_2(\phi),h_1;x,a]\times\delta[h_2(\phi_0)-h_2] \; . \tag{5.21}$$

It is understood in Eq. (5.21) that the two points needed to define $P(h_2,h_1,x)$ are the origin and (x,ϕ_0) in polar coordinates of (5.18). The angle ϕ_0 is arbitrary. Exponentiating the δ function and performing Gaussian integration one finds [cf. (2.4)] the two-point probability

$$P(h_2,h_1,x)=\frac{1}{\sqrt{4D\ln(x/a)}}\exp\left[-\frac{\pi(h_2-h_1)^2}{4D\ln(x/a)}\right] , \tag{5.22}$$

with the obvious normalizing prefactor. This function obeys a diffusionlike partial differential equation,

$$\partial_x P=\frac{D}{\pi x}\partial_{hh}P \; , \tag{5.23}$$

with a cutoff at small x, $x\geq a$. Equation (5.23) describes a logarithmically wandering interface [15].

VI. THE DIFFUSION EQUATION: THE BOUNDED RANDOM INITIAL CONDITION IN TWO DIMENSIONS

In this section we derive the exponents for the kinetic-energy decay in two dimensions and obtain the behavior resembling the $O(2)$ nonlinear σ model after the onset of bounded properties in the initial condition. The method of mirror images is again useful. Due to translational invariance and isotropy of the problem the only correlator that we need is $\langle h(x)h(0)\rangle$. The probability $P(h_2,h_1,x)$ given by Eq. (5.22) is sufficient to perform the calculation if used instead of $G(h,x)$, Eq. (4.6). Function P can be obtained from G by replacements $D\to D/\pi$ and $x\to\ln(x/a)$. The correlator (4.8) takes the form

$$\langle h_0(x)h_0(0)\rangle=\frac{32H^2}{\pi^4}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}\left[\frac{x}{a}\right]^{-\pi D(2n+1)^2/4H^2} . \tag{6.1}$$

Integration in (4.2) with the low limit being $x=a$ yields

$$E(t)=\frac{8\lambda^2H^2}{\pi^4\nu t}g\left[\frac{8\nu t}{a^2}\right] , \tag{6.2}$$

with the function

$$g(x)=\sum_{m=0}^{\infty}\frac{x^{-s(m)}}{(2m+1)^4}\{\Gamma[1-s(m),x^{-1}]-\Gamma[2-s(m),x^{-1}]\} \tag{6.3}$$

and $s(m)=\pi D(2m+1)^2/8H^2$. $\Gamma(a,b)$ is the incomplete gamma function. The limit $\nu t\gg a^2$ appears to be rich and is studied in Appendix B. Following Appendix B consider the following cases.

(i) $D/H^2<8/\pi$. Using (B1) we get

$$E(t)=\frac{\lambda^2 D}{\pi^3\nu t}\left[\frac{8\nu t}{a^2}\right]^{-\pi D/8H^2}\Gamma\left[1-\frac{\pi D}{8H^2}\right] \propto t^{-1-\pi D/8H^2} \; . \tag{6.4}$$

The decay exponent depends linearly upon the parameter D/H^2 and changes from -1 (cf. nonbounded case) down to -2 at the critical value $D/H^2=8/\pi$. Note that the corrections to Eq. (6.4) are of the order of $t^{-1-s(m)}$ so that there is an accumulation point at $D/H^2=0$. To get the prefactor in the limit $D/H^2\to 0$ [which is, of course, given by (3.6)] it is easier to return to the integral (4.2). The following formula is helpful

$$\int_0^\infty d\xi(\xi-1)e^{-\xi}\ln\xi=1 \; .$$

(ii) $D/H^2=8/\pi$. There is a logarithmic correction in the leading order

$$E(t)=\left[\frac{\lambda a H}{\pi^2\nu t}\right]^2\ln\left[\frac{8\nu t}{a^2}\right] . \tag{6.5}$$

(iii) $D/H^2>8/\pi$. Using (B3) and (B4) one finds

$$E(t)=\frac{1}{4\pi^3}\left[\frac{\lambda a H^2}{\nu t}\right]^2\left[s(0)\tan\left[\frac{\pi}{2\sqrt{s(0)}}\right]-\frac{\pi s(0)}{2}-\frac{\pi^3}{24}\right]\propto t^{-2}\ . \qquad (6.6)$$

We note that the dependence of the decay exponent on the parameter D/H^2 is similar to the $O(2)$ nonlinear σ model [10,11]. This similarity is not complete, however. Recall that the spin direction, which distribution is described by Q_2, is a *cyclic* variable, like our mirror imaged field h_0. At nonzero temperature (in our case, at nonzero D) there exist vortex excitations which interact like a Coulomb gas. For our field h_0 this would mean the presence of point defects in the vicinity of which $h_0(y,\phi)=2H\phi$. The velocity field $\mathbf{v}(y,\phi)=-(\lambda/y)\mathbf{e}_\phi$ is not curl-free. Recall now that the absence of vorticity was assumed by Burgers when he "derived" his equation from the NS equation. To make a consistent comparison with the Burgers equation we avoid vorticity at all stages. Other applications such as nonlinear heat flow and interface growth also do not lead to point defects under usual circumstances. Consequently, our correlators behave as low-temperature correlators in the σ model [10,11]. To show this, one may use the probability $P(h_2,h_1,x)$ to calculate the average $\langle\cos(h_2-h_1)\rangle$ usually considered in spin models. It is given by

$$\langle\cos(h_2-h_1)\rangle =\frac{1}{2}\left[\frac{x}{a}\right]^{-\pi D/H^2}+\sum_{n=0}^{\infty}\frac{8(2n+1)^2}{\pi^2[(2n+1)^2-4]^2}\times\left[\frac{x}{a}\right]^{-\pi D(2n+1)^2/4H^2} \underset{x\to\infty}{\longrightarrow}\frac{8}{9\pi^2}\left[\frac{x}{a}\right]^{-\pi D/4H^2}\ . \qquad (6.7)$$

This correlator cannot diminish steeper than x^{-2} (see Ref. [11]) and therefore $D/H^2=8/\pi$ is the largest value of D/H^2 when (6.7) (and the distribution Q_2) represents the spin model. This is precisely the transition value of D/H^2 found above. It is eight times larger than the Kosterlitz-Thouless transition temperature [11] (to indicate the connection, one assumes that $k_BT/J=2D$ and $H=\pi/2$, so that $k_BT_c/J=\pi/2$ corresponds to $D_c/H^2=1/\pi$).

In this respect it may seem surprising to find the abrupt transition at $D^2/H^2=8/\pi$. The analog of the steepest correlator (6.7) in our study is the kinetic-energy density of the *discontinuous* bounded distribution which diffusively decays as t^{-2}. The relaxation of continuous fields should not be faster. Thus the kinetic energy is a curious quantity which undergoes a transition with mirror images in the absence of topological charges. The transition is a special property of the sharp features of the potential associated with the Q_2 distribution. To clarify this one may calculate the kinetic-energy decay for the Gaussian distribution

$$Q_3\sim\exp\left[-\frac{1}{4D}\int d^dx\,[(\nabla h)^2+m^2h^2]\right] \qquad (6.8)$$

and find that there is a single crossover time at $t=1/\nu m^2$, which is D and d independent. The energy is given by

$$E(t)=\frac{\lambda^2Ddm^d}{2(4\pi)^{d/2}}e^{2m^2\nu t}\Gamma\left[-\frac{d}{2},2m^2\nu t\right]\ . \qquad (6.9)$$

At short times one observes the nonbounded decay (3.6) and after the crossover the decay reaches its maximum rate $t^{-d/2-1}$ without any unusual behavior in two dimensions. Thus the onset of bounded behavior is sensitive to the shape of the potential which provides the boundary: the parabolic potential of the distribution (6.8) leads to a behavior different from that of infinite well potential Q_2.

Returning to vortices, we note that diffusion equation and (formally) the Burgers equation may be considered with random initial conditions having a solenoidal component. We are not aware of possible physical applications, though. It is interesting to mention that for both equations the decay of vorticity is purely diffusional, while the potential component may have rich behavior since it is coupled to the solenoidal component.

VII. BURGERS TURBULENCE: THE BOUNDED RANDOM INITIAL DISTRIBUTION IN ONE DIMENSION

We have completed our preliminary steps of deriving the reference results for diffusion equation and begin to study the Burgers equation in the form of the field theory reproduced in Sec. II. The case of the nonbounded initial condition has been considered in Ref. [1]. We denote as $\Phi(h,y)$ the density of the function ψ (1.5) along the h axis; it is a sum over all the paths terminating at the point h for a given time y. This function obeys the forward and backward Schrödinger equation [1] (diffusion equation with a decay term)

$$\partial\Phi(h,y)/\partial y=\pm D\partial\Phi/\partial h^2\mp ug(y)e^{\lambda h/2\nu}\Phi\ , \qquad (7.1)$$

which is considered in the strip $|h|\le H$ (cf. Sec. IV). The boundary conditions are $\partial_h\Phi(\pm H,y)=0$ and the "initial" condition reflects the uniform distribution in the strip far from the origin $y=0$, i.e., far from the potential support (decay support): $\Phi(h,\mp\infty)=1/2H$.

Equation (7.1) can be simplified in the limit of zero viscosity $\nu\to 0$. Introducing the variable $h_u=-(2\nu/\lambda)\ln u$ one can rewrite (1.4) as

$$\langle h\rangle=\int_{-\infty}^{\infty}dh_u[\exp(-e^{-\lambda h_u/2\nu})-\psi(h_u)] \underset{\nu\to 0}{\longrightarrow}\int_{-\infty}^{\infty}dh_u[\Theta(h_u)-\psi(h_u)]\ , \qquad (7.2)$$

where $\Theta(z)$ is the step function. The decay term in Eq. (7.1) becomes a δ function, i.e., there appears an absorbing curve

$$h=h_u+y^2/2\lambda t \qquad (7.3)$$

and a pure diffusion equation

$$\partial\Phi(h,y)/\partial y=\pm D\partial^2\Phi/\partial h^2 \tag{7.4}$$

below this parabolic curve. We can see now that the Burgers approach [2] is equivalent to the present study in the limit $\nu\to 0$.

For some time there is no difference between bounded and unbounded cases since local properties of the distributions Q_1 and Q_2 are almost identical. In the limit of strong turbulence that we are interested in, $t \gg \nu^3/D^2\lambda^4$, the energy decays as [2,1]

$$E=\eta\left[\frac{D\lambda^2}{t}\right]^{2/3}, \tag{7.5}$$

where η is a number [16]. The averaged height of the corresponding interface problem grows as $\langle h\rangle=3\eta(D^2\lambda t)^{1/3}$. This regime ends when the averaged height approaches the limiting value H, since we have $\langle h\rangle<H$ by the above-mentioned boundary conditions. At later times the parabola (7.2) is quite close to the upper boundary H. One can then approximate the parabola by making the boundary $h=H$ absorbing within the region $|y|<y_0$, where

$$y_0=\sqrt{2\lambda t(H-h_u)}\ ; \tag{7.6}$$

see (7.2). Selecting the forward version of the diffusion approach (7.4), we start with the function $\Phi(h)$ uniform in $[-H,H]$ at the point $y=-y_0$. The solution at the end of the absorbing region $y=+y_0$ is given by

$$\Phi(h,y)=\int_{-H}^{H}dh'\frac{1}{\sqrt{8\pi D y_0}}\sum_{m=-\infty}^{\infty}\left[\exp\left[-\frac{(h-h'+4Hm)^2}{8Dy_0}\right]-\exp\left[-\frac{(h+h'-2H+4Hm)^2}{8Dy_0}\right]\right]. \tag{7.7}$$

The minus sign here [cf. (4.5)] accounts for the absorbing boundary. It follows from (7.2) that to find the averaged height $\langle h\rangle$ one has to evaluate the integral

$$\langle h\rangle=\int_0^H dh_u[1-\int_{-H}^{H}dh\,\Phi(h,y)]=H-\int_0^H\int_{-H}^{H}dh_u dh\,\Phi(h,y)\ . \tag{7.8}$$

Using another representation (A1) of the elliptic functions entering (7.7) one obtains that $H-\langle h\rangle$ equals

$$\int_0^H\int_{-H}^{H}\int_{-H}^{H}dh_u dh\,dh'\sum_{n=1}^{\infty}\exp\left[-\frac{\pi^2n^2Dy_0}{2H^2}\right]\left\{\cos\left[\frac{\pi n}{2H}(h-h')\right]-\cos\left[\frac{\pi n}{2H}(h+h'-2H)\right]\right\}$$
$$=\frac{8}{\pi^2}\int_0^H dh_u\sum_{l=0}^{\infty}\frac{\exp\left[-\frac{\pi^2Dy_0(2l+1)^2}{2H^2}\right]}{(2l+1)^2}\ . \tag{7.9}$$

With the help of (7.6) one gets the late-time asymptotic behavior for the averaged height

$$\langle h\rangle=H-\frac{H^4}{30D^2\lambda t} \tag{7.10}$$

and the kinetic-energy decay

$$E=\frac{H^4}{30D^2t^2}\ . \tag{7.11}$$

This is a very fast decay for one-dimensional problem, faster than the asymptotic behavior of pure diffusion (4.9). One then expects a second crossover to pure diffusion if the finite viscosity is allowed back into Eq. (7.1). The diffusive behavior at late times must be accessible by direct perturbation in the decay strength. Let us introduce an auxiliary function

$$\Psi(y)=\int_{-H}^{H}\Phi(h,y)dh,\quad \Psi(\infty)=\psi\ . \tag{7.12}$$

To zeroth order in u we have $\Phi_0(h,y)=1/2H$; this is essentially an adiabatic approximation. Decay influences the *amplitude* of the solution; to the first order

$$\Phi(h,y)=\Psi_1(y)/2H\ . \tag{7.13}$$

Function (7.13) upon substitution into Eq. (7.1) and integrating over $[-H,H]$ has the form

$$\Psi_1(y)=\exp\left[-u\frac{\sinh\kappa}{\kappa}\int_{-\infty}^{y}dy\,g(y)\right], \tag{7.14}$$

where the Reynolds number is defined as $\kappa=\lambda H/2\nu$. As usual the knowledge of the solution to zeroth order is enough to calculate the leading behavior of the integrated properties. Returning to the averaged field h, which is given by Eqs. (1.4), (7.12), and, to this order, (7.14), we get

$$\langle h\rangle=\frac{2\nu}{\lambda}\ln\left[\frac{\sinh\kappa}{\kappa}\right]. \tag{7.15}$$

In terms of (1.2) the formula (7.15) implies that the averaged height ceases to move at late times and saturates at the level (7.15) below H. We again expect from this result that the final stage of decay is pure diffusion, since saturation means small gradients and irrelevance of the λ term in (1.2).

One has to determine the correct amplitude of the diffusionlike decay $E(t)\propto t^{-3/2}$ (see Sec. IV for the derivation of this power law). To go beyond zeroth order in u regarding Eq. (7.1) we make use of the expansion

$$\Phi(h,y)=\frac{\Psi_1(y)}{2H}\left[1+u\mu_1(h,y)+\frac{u^2}{2}\mu_2(h,y)+\cdots\right], \tag{7.16}$$

which is inspired by the WKB approximation. Substituting (7.16) into Eq. (7.1) and collecting similar terms we get the equation for first order in u corrections to the solution

$$\partial_y\mu_1-D\partial_{hh}\mu_1=g(y)\left[\frac{\sinh\kappa}{\kappa}-e^{\lambda h/2\nu}\right]. \tag{7.17}$$

Its solution is

$$\mu_1(h,y)=\int_{-\infty}^{y}dy'\int_{-H}^{H}dh'g(y')\left[\frac{\sinh\kappa}{\kappa}-e^{\lambda h'/2\nu}\right]\times G_\Sigma(h,h',y-y'), \tag{7.18}$$

where the appropriate Green's function $\mathcal{G}$ is defined by (4.5). It is easy to check that

$$\int_{-H}^{H}dh\,\mu_1(h,y)=0, \tag{7.19}$$

so that there is no contribution to the function ψ in this order. Second order in u gives the equation

$$\partial_y\mu_2-D\partial_{hh}\mu_2=g(y)\mu_1\left[\frac{\sinh\kappa}{\kappa}-e^{\lambda h/2\nu}\right]. \tag{7.20}$$

Integrating this equation with the help of G_Σ we may represent the quantity of interest $I=\int_{-H}^{H}dh\,\mu_2(h,\infty)$ in the form

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{y_2}dy_2dy_1g(y_1,t)g(y_2,t)\times\int_{-H}^{H}\int_{-H}^{H}dh_1dh_2Z(h_1)Z(h_2)\times G_\Sigma(h_2,h_1,y_2-y_1), \tag{7.21}$$

where

$$Z(h)=\sinh(\kappa)/\kappa-\exp(\lambda h/2\nu). \tag{7.22}$$

The evaluation of the integral (7.21) follows the same lines as (4.7). We first change variables $y=y_2-y_1$ and $z=y_2+y_1$ and integrate out z. One finds

$$I=\frac{1}{\pi\sqrt{8D\nu t}}\int_0^{\infty}\frac{dy}{\sqrt{y}}e^{-y^2/8\nu t}J(y), \tag{7.23}$$

where

$$J(y)=\int_{-H}^{H}\int_{-H}^{H}dh_1dh_2Z(h_1)Z(h_2)\times\sum_{m=-\infty}^{\infty}\left[\exp\left[-\frac{(h_2-h_1+4Hm)^2}{4Dy}\right]+\exp\left[-\frac{(h_2+h_1-2H+4Hm)^2}{4Dy}\right]\right]. \tag{7.24}$$

Changing variables and using symmetry properties of the integrand we obtain

$$J(y)=4H^2\int_0^1d\eta\,p(\eta)\sum_{m=-\infty}^{\infty}\exp\left[-\frac{H^2}{Dy}(\eta+2m)^2\right], \tag{7.25}$$

with

$$\begin{aligned}p(\eta)=&\int_{-1+\eta}^{1-\eta}d\eta'Z[H(\eta'-\eta)]Z[H(\eta'+\eta)]\\&+\int_0^{\eta}d\eta'\{\tfrac{1}{2}Z[H(\eta-\eta'-1)]Z[H(\eta+\eta'-1)]+\tfrac{1}{2}Z[H(1-\eta-\eta')]Z[H(1-\eta+\eta')]\}\\&=\tfrac{1}{4}\kappa^{-2}e^{-2\kappa(1+\eta)}(e^{2\kappa}-e^{4\kappa}-e^{2\kappa\eta}-e^{4\kappa\eta}+2e^{2\kappa(1+\eta)}-e^{2\kappa(2+\eta)}\\&\quad+e^{2\kappa(1+2\eta)}+2\kappa e^{4\kappa}-2\kappa e^{4\kappa\eta}-\eta e^{2\kappa\eta}+2\eta e^{2\kappa(1+\eta)}-\eta e^{2\kappa(2+\eta)}+2\kappa^2\eta e^{4\kappa}+2\kappa^2\eta e^{4\kappa\eta}).\end{aligned} \tag{7.26}$$

Note that $\int_0^1d\eta\,p(\eta)=0$. Now the sum entering (7.25) can be rewritten using (A1). Denoting $p_l=\int_0^1d\eta\,p(\eta)\cos(\pi l\eta)$ and integrating over y we find

$$I=4H\sum_{l=1}^{\infty}p_l\exp\left[2\nu t\left[\frac{\pi^2l^2D}{4H^2}\right]^2\right]\operatorname{erfc}\left[\sqrt{2\nu t}\,\frac{\pi^2l^2D}{4H^2}\right]. \tag{7.27}$$

The applied expansion (7.16) is consistent when D diffusion in the h direction is faster than ν diffusion in the y direction, i.e., $D^2\nu t/H^4\gg1$. Using the asymptotic form of the error function and returning to kinetic energy, we get

$$E(t)=2\nu\partial_t\int_0^{\infty}\frac{du}{u}\left[e^{-u}-\Psi_0-\frac{u^2}{2}\frac{1}{2H}\Psi_0I\right]=\frac{4\nu H^2}{\pi^2\sqrt{2\pi\nu t}\,Dt}\frac{\kappa^2}{\sinh^2\kappa}\sum_{l=1}^{\infty}\frac{p_l}{l^2}. \tag{7.28}$$

The remaining sum can be evaluated by using the definition of p_l and (7.26),

$$\sum_{l=1}^{\infty}\frac{p_l}{l^2}=\frac{\pi^2}{192\kappa^4e^{2\kappa}}(21-42e^{2\kappa}+21e^{4\kappa}+30\kappa-30\kappa e^{4\kappa}+13\kappa^2+10\kappa^2e^{2\kappa}+13\kappa^2e^{4\kappa}). \tag{7.29}$$

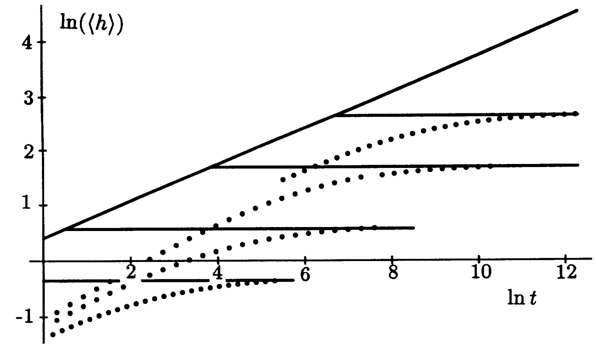

FIG. 1. Log_e-$\log_e$ plot of $\langle h \rangle$ vs time obtained by numerical integration of Eq. (7.1) and using (1.4) and (7.12). The tilted line is the Burgers asymptotic dependence $\langle h \rangle = 3\eta D^{1/3}(\lambda t)^{1/3}$. The horizontal lines are the saturation heights, given by Eq. (7.15), and the corresponding heights used in simulation are $H = 3, 5, 10, 20$ from bottom to top. Other parameters used are $D = \nu = \lambda = 1$. Larger values of H require too prolonged simulations.

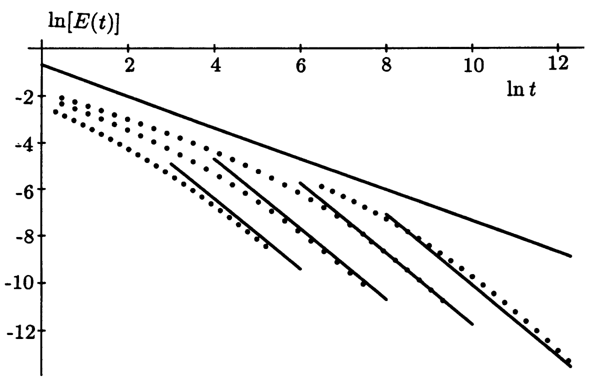

FIG. 2. Log_e-$\log_e$ plot of kinetic energy E vs time obtained by using $E = \lambda\partial_t\langle h \rangle$ and the data of Fig. 1. The long tilted line is the Burgers asymptotic dependence $E = \eta D^{1/3}\lambda^{4/3}t^{-2/3}$. Other lines are the asymptotic dependences given by Eq. (7.29). At these H's one cannot use Eq. (7.30) yet since the limit of strong turbulence is not achieved. The transient $E \sim t^{-2}$ behavior just starts to appear; one may see only an indication of this in the form of a kinklike bend on the curve with $H = 20$.

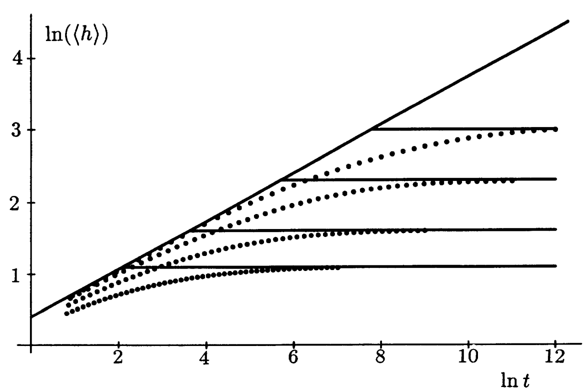

FIG. 3. Log_e-$\log_e$ plot of $\langle h \rangle$ vs time obtained by numerical integration of Eq. (7.4) and absorbing line (7.2) (this is the inviscid limit). Solid lines are the Burgers asymptotic dependence and saturation values $\langle h \rangle = H$. The corresponding heights used in simulation are $H = 3, 5, 10, 20$ from bottom to top. Other parameters used are $D = 4$ and $\lambda = 1$.

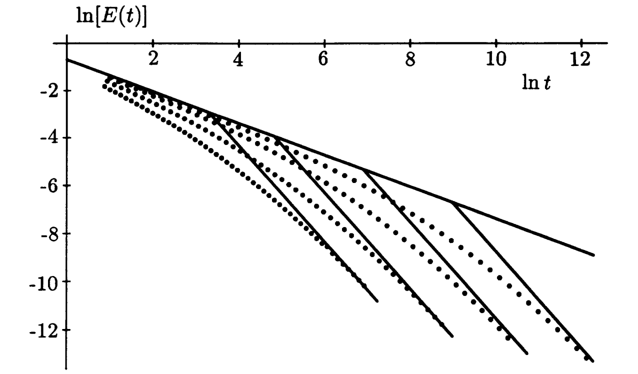

FIG. 4. Log_e-$\log_e$ plot of kinetic energy E vs time obtained by using $E = \lambda\partial_t\langle h \rangle$ and the data of Fig. 3. The long tilted line is the Burgers asymptotic dependence and other lines are the asymptotic dependences given by Eq. (7.11).

The asymptotical form in the limit of strong turbulence $\kappa \gg 1$ is

$$E(t) = \frac{13\nu^2 H^2}{48\sqrt{2\pi}D(\nu t)^{3/2}}, \tag{7.30}$$

a diffusive decay which differs from (4.9) only by its amplitude.

We may now reconstruct the entire decay process. At short times, when degeneracy of the turbulence is not important the decay is given by the Burgers formula (7.5). The averaged height $\langle h \rangle$ approaches H at $t_1 = H^3/D^2\lambda$. From the other side the diffusive decay (7.30) applies at $D^2\nu t/H^4 \gg 1$, when the ν diffusion length $(\nu t)^{1/2}$ exceeds H^2/D. This defines $t_2 = H^4/D^2\nu$, the second crossover time. The ratio of the two times is $t_2/t_1 \sim \kappa$. In the case of strong turbulence $\kappa \gg 1$, these results indicate the existence of an intermediate regime at $t_1 \ll t \ll t_2$ (if $\kappa \ll 1$, the entire process is just diffusion). The intermediate regime is described by (7.11). We have made numerical simulations of Eqs. (7.1) and (7.4). The results are shown in Figs. 1–4.

VIII. BURGERS TURBULENCE IN TWO DIMENSIONS

Using the results of Sec. V we introduce the density of the functional integral (1.6) and by analogy with the preceding section we write the functional diffusion (Schrödinger) equation, which this density satisfies,

$$\partial_y \Phi[h(\phi), y] = \pm\hat{D}\Phi \mp uyg(y)\Phi \int_0^{2\pi} d\phi\, e^{\lambda h(\phi)/2\nu}. \tag{8.1}$$

The equation is considered on the space of all $h(\phi)$ profiles and radius time y exceeds the cutoff $y > a$. The derivation of the operator $\hat{D}$ implies that Φ is a double density, with specified initial and final profiles.

We shall only consider the inviscid limit for the nonbounded case Q_1. Even in the limit of vanishing viscosity ν the ratio $\nu t/a^2$ should be assumed to be large. The structure of Eq. (8.1) suggests the change of the time variable $z = \ln(y/\sqrt{4\nu t})$. Equation (8.1) can be rewritten as

$$\partial_z \Phi = D \int_0^{2\pi} d\phi \frac{\delta^2 \Phi}{\delta h^2} - \frac{\Phi}{4D} \int_0^{2\pi} d\phi \left[\frac{dh}{d\phi} \right]^2 - 2\Phi \int_0^{2\pi} d\phi \exp \left[2z - e^{2z} + \frac{\lambda [h(\phi) - h_u]}{2\nu} \right] , \tag{8.2}$$

where we selected the forward version and introduced the familiar parameter $h_u = -(2\nu/\lambda)\ln u$ (see Sec. VII). Equation (8.2) exhibits two decay terms. One of them comes from the diffusion of strings and the other is generated by the interaction term in (1.7). By analogy to Sec. VII, we define a surface

$$h(\phi,z) = h_u + \frac{2\nu}{\lambda}(e^{2z} - 2z) , \tag{8.3}$$

above which the interaction-generated decay dominates. When $\nu \rightarrow 0$ the surface (8.3) becomes a plane parallel to the $\{y,\phi\}$ plane up to the region of large positive z when the exponential term in (8.3) can no longer be neglected. Let us denote these values of z as $\bar{z}$. Below the absorbing plane $h = h_u$ we have the pure diffusion equation (5.17) with the point source at $(h,z) = (0, z_{\min})$, $z_{\min} = \ln(a/\sqrt{\nu t})$. The absorbing surface can be accounted by mirror imaging of this source with opposite sign. Thus the solution for Φ reads

$$\Phi[h(\phi),z] = \mathcal{F}\left[h(\phi), 0; \frac{e^{\bar{z}}}{4\nu t}, a \right] - \mathcal{F}\left[h(\phi), 2h_u; \frac{e^{\bar{z}}}{4\nu t}, 0 \right] \tag{8.4}$$

with the functional (5.18). This solution has to be integrated over all final profiles $h(\phi)$ (or sections of these) which end below the absorbing surface (8.3). It can be conveniently done in two steps. First, we integrate over all profiles which pass through a specified point $(\bar{z}, \phi_0)$; this leads to the two-point probability P, introduced in Sec. V. With the help of (7.2) we perform the second step

$$\begin{aligned} \langle h \rangle &= \int_0^\infty dh_u \left\{ 1 - \int_{-\infty}^{h_u} dh \left[P\left[h, 0, \frac{e^{\bar{z} - z_{\min}}}{4\nu t} \right] - P\left[h, 2h_u, \frac{e^{\bar{z} - z_{\min}}}{4\nu t} \right] \right] \right\} \\ &= \int_0^\infty dh_u \, \mathrm{erfc}\left[\frac{\pi^{1/2} h_u}{4D(\bar{z} - z_{\min})} \right] = \frac{1}{\pi}\sqrt{4D(\bar{z} - z_{\min})} . \end{aligned} \tag{8.5}$$

From the definition of $\bar{z}$ it is clear that its time dependence is of the type $\bar{z} = \frac{1}{2}\ln(\lambda\langle h \rangle/2\nu) \sim \ln\ln t$. With logarithmic precision one can set $\bar{z} = 0$ and obtain

$$\langle h \rangle = \left[\frac{2D}{\pi^2} \ln\left[\frac{\nu t}{a^2} \right] \right]^{1/2} , \tag{8.6}$$

$$E = \frac{\lambda D^{1/2}}{\pi t \sqrt{2\ln(\nu t/a^2)}} . \tag{8.7}$$

The analogous time dependence $t^{-1}\ln^{-1/2} t$ is obtained in Ref. [4] for a different problem.

We briefly discuss the bounded case. As in one dimension there is a crossover time which can be found by equating $\langle h \rangle = H$, see Eq. (8.6), $t_1 = (a^2/\nu)e^{\pi^2 H^2/2D}$. In the inviscid limit this is the end of the evolution. If ν is finite, kinetic energy quickly falls down and later is of order $(a/t)^2$.

CONCLUSION

We have presented a detailed study of Burgers turbulence decay in one and two dimensions and we think that the decay of kinetic energy is more or less understood. In one dimension our realization of the degenerate turbulence indicates the existence of three-stage relaxation which serves as a counterexample for the prediction of the similarity hypothesis based upon the zero Loitsyanskii correlator. Obviously, the zero correlator does not fully define the distribution and the existence of other counterexamples is expected. The results $d=2$ demonstrate an interesting behavior when the energy decay contains logarithmic factors and an explicit cutoff dependence. Therefore, the scaling (if it exists) is preceded by an exponentially long transient required to ensure $\ln(\nu t/a^2) \gg 1$. The existence of an exponentially slow crossover has been recognized earlier in the noise-driven case [17]. The crossover complicates the physics of the noise-driven case, where exponents obtained by numerical simulations are found to be model dependent. We hope that this paper will stimulate numerical and RG studies of the initial condition problems. Subsequently, the exact results from this method, computer simulations, RG analysis, and, hopefully, the application of the direct integration approximation by Kraichnan will be instructively compared at some stage.

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APPENDIX A: TWO EXPANSIONS OF θ_3 ELLIPTIC FUNCTION

The following equality is useful in treating mirror images:

$$\sum_{m=-\infty}^{\infty} e^{-(\eta+2m)^2/x} = \frac{\sqrt{\pi x}}{2}\theta_3\left[\frac{\pi\eta}{2}, e^{-\pi^2 x/4}\right] = \frac{\sqrt{\pi x}}{2}\left[1+2\sum_{m=1}^{\infty} e^{-\pi^2 m^2 x/4}\cos(\pi m\eta)\right] .$$

APPENDIX B: FUNCTION $g(x)$ DEFINED BY (6.3)

Consider the limit $x \gg 1$. There is a special sequence $s(m)=1,2,3\ldots$ when logarithmic terms appear in the expansion of incomplete gamma function and different corrections to the main dependence acquire logarithms. One can identify three different situations.

(i) $s(0)<1$. We find

$$g(x)=s(0)x^{-s(0)}\Gamma(1-s(0)) . \tag{B1}$$

(ii) $s(0)=1$. There is a logarithmic correction in the leading order,

$$g(x)=\frac{\ln x}{x}-\left[\gamma+\frac{\pi^4}{96}\right]+O(x^{-2}) . \tag{B2}$$

(iii) $s(0)>1$. The leading dependence can be shown to be

$$g(x)=\frac{1}{x}\left[\frac{\pi}{4}s(0)\tan\left[\frac{\pi}{2\sqrt{s(0)}}\right]-\frac{\pi^2 s(0)}{8}-\frac{\pi^4}{96}\right] +O(x^{-s(0)})+O(x^{-2}) \tag{B3}$$

with the asymptotics

$$xg(x)=\frac{1}{s(0)-1}-\frac{\pi^4+12\pi^2-216}{96}+O(s(0)-1) , \quad s(0)-1\ll 1$$

$$xg(x)=\frac{\pi^6}{960s(0)}+\frac{17\pi^8}{2^6 15 s(0)^2}+O(s(0)^{-3}) , \quad s(0)\gg 1 . \tag{B4}$$

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